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Asymptotic stability of the solution of the $M/M^B/1$ queueing model

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Abstract

By using positive C_0-semigroup theory we study the asymptotic stability of the solution of a bulk queueing model $M/M^B/1$.

Keywords: $M/M^B/1$ queueing model; Positive C_0-semigroup; Dirichlet operator; Resolvent set; Eigenvalue

1. Introduction

The $M/M^B/1$ model is one important model in operations research. Many authors are studying it (see [1–4]). According to [1], The $M/M^B/1$ queueing model can be expressed as:

$$\frac{\mathrm{d}p_{0,0}(t)}{\mathrm{d}t} = -\lambda p_{0,0}(t) + \mu \int_0^\infty p_{0,1}(x,t)\mathrm{d}x, \tag{1}$$

$$\frac{\partial p_{0,1}(x,t)}{\partial t} + \frac{\partial p_{0,1}(x,t)}{\partial x} = -(\lambda+\mu)p_{0,1}(x,t), \tag{2}$$

$$\frac{\partial p_{n,1}(x,t)}{\partial t} + \frac{\partial p_{n,1}(x,t)}{\partial x} = -(\lambda+\mu)p_{n,1}(x,t) + \lambda p_{n-1,1}(x,t), \quad n \geq 1, \tag{3}$$

$$p_{0,1}(0,t) = \sum_{k=1}^{B} \mu \int_0^\infty p_{k,1}(x,t)\mathrm{d}x + \lambda p_{0,0}(t), \tag{4}$$

$$p_{n,1}(0,t) = \mu \int_0^\infty p_{n+B,1}(x,t)\mathrm{d}x, \quad n \geq 1, \tag{5}$$

$$p_{0,0}(0) = 1, \qquad p_{n,1}(x,0) = 0, \quad n \geq 0. \tag{6}$$

Here $(x,t) \in [0,\infty) \times [0,\infty)$, $p_{0,0}(t)$ represents the probability that the queue [not system (queue + service)] is empty and the server is idle at time t, $p_{n,1}(x,t)\mathrm{d}x$ represents the probability that at time t there are n customers in the

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queue [not in system (queue+service)] and the elapsed service time lies in $(x, x + \mathrm{d}x]$, μ is the mean service rate of the server, λ is the mean arrival rate of the customer, B represents the maximum size of service.

The book [1] established the mathematical model of the $M/M^B/1$ queue and studied the static solution by using probability generating functions under following hypothesis:

Hypothesis 1. The $M/M^B/1$ queueing model has a unique positive time-dependent solution $p(x, t)$.

Hypothesis 2. The time-dependent solution $p(x, t)$ converges to the static solution $p(x)$ as time tends to infinite.

Here

$$p(x,t) = (p_{0,0}(t), p_{0,1}(x,t), p_{1,1}(x,t), p_{2,1}(x,t), p_{3,1}(x,t), \ldots),$$
$$p(x) = (p_{0,0}, p_{0,1}(x), p_{1,1}(x), p_{2,1}(x), p_{3,1}(x), \ldots).$$

In [2], Gupur converted this model into an abstract Cauchy problem on a suitable Banach space and then proved the existence of a unique positive time-dependent solution by using the theory of C_0-semigroups of linear operators. In other words, the author proved that the above Hypothesis 1 holds. In [3], Gupur proved that all points on the imaginary axis except for 0 belong to the resolvent set of the $M/M^B/1$ operator A, but until now the proof of the Hypothesis 2 has been still an open problem (see [3]).

In this paper, first we write Dirichlet operator; second by using Dirichlet operator we prove that 0 is an eigenvalue of the $M/M^B/1$ operator A; and third, we prove that all points on the imaginary axis except for 0 belong to the resolvent set of the $M/M^B/1$ operator A; fourth we show that the semigroup $T(t)$ generated by A is irreducible; finally, by using those results and Theorem 2.1 we prove that the Hypothesis 2 holds.

2. The problem as an abstract Cauchy problem

We first reformulate the system (1)–(6) an abstract Cauchy problem with an operator $(A, D(A))$ on a suitable stated space. The stated space X is chosen as

$$X = \left\{ p \in \mathbb{C} \times L^1[0,\infty) \times L^1[0,\infty) \times \cdots \;\middle|\; \|p\| = |p_{0,0}| + \sum_{n=0}^{\infty} \|p_{n,1}\|_{L^1[0,\infty)} < \infty \right\}.$$

It is obvious that X is a Banach space.

To define the operator $(A, D(A))$ we introduce a maximal operator $(A_m, D(A_m))$ on X as

$$A_m = \begin{pmatrix} -\lambda & \mu\psi & 0 & 0 & \cdots \\ 0 & D & 0 & 0 & \cdots \\ 0 & \lambda & D & 0 & \cdots \\ 0 & 0 & \lambda & D & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

$$D(A_m) = \left\{ p \in X \;\middle|\; \frac{\mathrm{d}p_{n,1}(x)}{\mathrm{d}x} \in L^1[0,\infty),\ \ p_{n,1}(x) \text{ is absolutely continuous function } (n \geq 0) \right\}.$$

Here and in the following ψ denotes the linear functional:

$$\psi : L^1[0,\infty) \to \mathbb{C}, \qquad f \mapsto \psi(f) := \int_0^\infty f(x)\mathrm{d}x.$$

Moreover, the operator D on $W^{1,1}[0,\infty)$ are defined as

$$D := -\frac{\mathrm{d}}{\mathrm{d}x} - (\lambda\eta_1 + \mu).$$

To formulate the boundary conditions (4) and (5) we will use the following boundary operators L and Φ mapping into the boundary space

$$\partial X := l^1.$$

The operator L is defined as

$$L : D(A_m) \to \partial X, \qquad \begin{pmatrix} p_{0,0} \\ p_{0,1} \\ p_{1,1} \\ p_{2,1} \\ \vdots \end{pmatrix} \mapsto L \begin{pmatrix} p_{0,0} \\ p_{0,1} \\ p_{1,1} \\ p_{2,1} \\ \vdots \end{pmatrix} := \begin{pmatrix} p_{0,1}(0) \\ p_{1,1}(0) \\ p_{2,1}(0) \\ \vdots \end{pmatrix},$$

and the operator $\Phi \in \mathcal{L}(X, \partial X)$ is defined as

$$\Phi = \begin{pmatrix} \lambda & 0 & \overbrace{\mu\psi \quad \mu\psi \quad \cdots \quad \mu\psi}^{B} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 \quad\;\; 0 \quad\;\; \cdots \quad\;\; 0 & \mu\psi & 0 & 0 & \cdots \\ 0 & 0 & 0 \quad\;\; 0 \quad\;\; \cdots \quad\;\; 0 & 0 & \mu\psi & 0 & \cdots \\ \vdots & \vdots & \vdots \quad\;\; \vdots \quad\;\; \vdots \quad\;\; \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

The operator $(A, D(A))$ on X is given as

$$Ap := A_m p, \qquad D(A) := \{p \in D(A_m) \mid Lp = \Phi p\}.$$

The above Eqs. (1)–(6) are equivalent to the abstract Cauchy problem

$$\begin{cases} \dfrac{\mathrm{d}p(t)}{\mathrm{d}t} = Ap(t), \quad t \in [0, \infty), \\ p(0) = (1, 0, 0, \ldots,) \in X. \end{cases}$$

In [3], Gupur obtained the following result: when the traffic intensity ρ satisfies

$$\rho = \frac{\lambda}{\mu} < 1.$$

Theorem 2.1. *The operator $(A, D(A))$ generates a positive contraction C_0-semigroup $(T(t))_{t\geq 0}$.*

3. Main results

In this section we investigate the spectrum $\sigma(A)$ of A and then give our main result on the asymptotic behaviour of the solutions. We first characterize $\sigma(A)$ by the spectrum of an infinite scalar matrix, i.e. an operator on the boundary space ∂X. To do so we apply results from [5]. Therefore we need the operator $(A_0, D(A_0))$ defined by

$$D(A_0) := \{p \in D(A_m) | Lp = 0\},$$
$$A_0 p := A_m p.$$

By [5, Lemma 1.2] we can decompose $D(A_m)$ for any $\gamma \in \rho(A_0)$ as

$$D(A_m) = D(A_0) \oplus \ker(\gamma - A_m).$$

A simple calculation shows that $\ker(\gamma - A_m)$ has the form

$$\ker(\gamma - A_m) = \left\{ p(x) \in D(A_m) \;\middle|\; \begin{aligned} & p(x) = (p_{0,0}, p_{0,1}(x), p_{1,1}(x), p_{2,1}(x) \cdots); \\ & p_{0,0} = \frac{\mu c_1}{(\gamma + \lambda + \mu)(\gamma + \lambda)}; \\ & p_{n,1}(x) = \mathrm{e}^{-(\gamma+\lambda+\mu)x} \sum_{k=0}^{n} \frac{\lambda^k}{k!} x^k c_{n+1-k}, \quad n \geq 0; \\ & \quad \text{and} \quad (c_n)_{n\geq 1} \in l^1 \end{aligned} \right\}.$$

Moreover, since L is surjective, $L|_{\ker(\gamma-A_m)} : \ker(\gamma - A_m) \to \partial X$ is invertible for any $\gamma \in \rho(A_0)$, see [5, Lemma 1.2]. We denote its inverse by

$$D_\gamma := (L|_{\ker(\gamma-A_m)})^{-1} : \partial X \longrightarrow \ker(\gamma - A_m),$$

and D_γ will be called "Dirichlet operator".

We now give the explicit form of D_γ.

Lemma 3.1. *For every $\gamma \in \rho(A_0)$, the operator D_γ has the form*

$$D_\gamma = \begin{pmatrix} \frac{\mu}{(\gamma+\lambda)(\gamma+\lambda+\mu)} & 0 & 0 & 0 & \cdots \\ e^{-(\gamma+\lambda+\mu)x} & 0 & 0 & 0 & \cdots \\ \lambda x e^{-(\gamma+\lambda+\mu)x} & e^{-(\gamma+\lambda+\mu)x} & 0 & 0 & \cdots \\ \frac{\lambda^2}{2!}x^2 e^{-(\gamma+\lambda+\mu)x} & \lambda x e^{-(\gamma+\lambda+\mu)x} & e^{-(\gamma+\lambda+\mu)x} & 0 & \cdots \\ \frac{\lambda^3}{3!}x^3 e^{-(\gamma+\lambda+\mu)x} & \frac{\lambda^2}{2!}x^2 e^{-(\gamma+\lambda+\mu)x} & \lambda x e^{-(\gamma+\lambda+\mu)x} & e^{-(\gamma+\lambda+\mu)x} & \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

At first we prove the following lemmas in order to obtain the main result in this paper. With the help of the operators D_γ and Φ we now characterise the spectrum $\sigma(A)$ and the point spectrum $\sigma_p(A)$ of A:

Lemma 3.2. *Let $\gamma \in \rho(A_0)$. Then the following holds.*

(1) $\gamma \in \sigma(A) \iff 1 \in \sigma(D_\gamma \Phi) \iff 1 \in \sigma(\Phi D_\gamma)$.
(2) $\gamma \in \sigma_p(A) \iff 1 \in \sigma_p(D_\gamma \Phi) \iff 1 \in \sigma_p(\Phi D_\gamma)$.

Proof. By [5, Lemma 1.4], for any $\gamma \in \rho(A_0)$ we have

$$\gamma - A = (\gamma - A_0)(\mathrm{Id} - D_\gamma \Phi). \tag{7}$$

This shows the first equivalence in (1) and (2).

It follows from this that

$$\gamma - A \text{ is injective } \iff \mathrm{Id} - D_\gamma \Phi \text{ is injective},$$

hence

$$\gamma \in \sigma_p(A) \iff 1 \in \sigma_p(D_\gamma \Phi).$$

In the following we prove that

$$1 \in \sigma_p(D_\gamma \Phi) \iff 1 \in \sigma_p(\Phi D_\gamma).$$

If we suppose that $\mathrm{Id} - \Phi D_\gamma$ is injective, then there exists an operator B such that $B(\mathrm{Id} - \Phi D_\gamma) = \mathrm{Id}$. Since

$$\begin{aligned}(\mathrm{Id} + D_\gamma B \Phi)(\mathrm{Id} - D_\gamma \Phi) &= \mathrm{Id} - D_\gamma \Phi + D_\gamma B(\mathrm{Id} - \Phi D_\gamma)\Phi \\ &= \mathrm{Id} - D_\gamma \Phi + D_\gamma \Phi = \mathrm{Id},\end{aligned}$$

the operator $\mathrm{Id} - D_\gamma \Phi$ is injective.

If we suppose that $\mathrm{Id} - D_\gamma \Phi$ is injective, then there exists an operator B' such that $B'(\mathrm{Id} - \Phi) = \mathrm{Id}$. Since

$$\begin{aligned}(\mathrm{Id} + \Phi B' D_\gamma)(\mathrm{Id} - \Phi D_\gamma) &= \mathrm{Id} - \Phi D_\gamma + \Phi B'(\mathrm{Id} - D_\gamma \Phi) D_\gamma \\ &= \mathrm{Id} - \Phi D_\gamma + \Phi D_\gamma = \mathrm{Id},\end{aligned}$$

the operator $\mathrm{Id} - \Phi D_\gamma$ is injective. It follows from this that:

$$1 \in \sigma_p(D_\gamma \Phi) \iff 1 \in \sigma_p(\Phi D_\gamma).$$

A similar computation as above shows that for $1 \in \rho(D_\gamma \Phi)$ also

$$(\mathrm{Id} - \Phi D_\gamma)(\mathrm{Id} + \Phi B' D_\gamma) = \mathrm{Id}$$

holds and for $1 \in \rho(\Phi D_\gamma)$

$$(\mathrm{Id} - D_\gamma \Phi)(\mathrm{Id} + D_\gamma B \Phi) = \mathrm{Id}.$$

Hence,

$$1 \in \sigma(D_\gamma \Phi) \iff 1 \in \sigma(\Phi D_\gamma). \quad \square$$

Remark 3.3. For $\gamma \in \rho(A_0)$ the operator ΦD_γ is represented by the following matrix:

$$\Phi D_\gamma = \begin{pmatrix} \frac{\mu\lambda}{(\lambda+\mu)\Gamma} + \sum_{k=1}^{B} \frac{\mu\lambda^k}{\Gamma^{k+1}} & \sum_{k=0}^{B-1} \frac{\mu\lambda^k}{\Gamma^{k+1}} & \sum_{k=0}^{B-2} \frac{\mu\lambda^k}{\Gamma^{k+1}} & \cdots & \frac{\mu}{\Gamma} + \frac{\mu\lambda}{\Gamma^2} & \frac{\mu}{\Gamma} & 0 & 0 & \cdots \\ \frac{\mu\lambda^{B+1}}{\Gamma^{B+2}} & \frac{\mu\lambda^{B}}{\Gamma^{B+1}} & \frac{\mu\lambda^{B-1}}{\Gamma^{B}} & \cdots & \frac{\mu\lambda^2}{\Gamma^3} & \frac{\mu\lambda}{\Gamma^2} & \frac{\mu}{\Gamma} & 0 & \cdots \\ \frac{\mu\lambda^{B+2}}{\Gamma^{B+3}} & \frac{\mu\lambda^{B+1}}{\Gamma^{B+2}} & \frac{\mu\lambda^{B}}{\Gamma^{B+1}} & \cdots & \frac{\mu\lambda^3}{\Gamma^4} & \frac{\mu\lambda^2}{\Gamma^3} & \frac{\mu\lambda}{\Gamma^2} & \frac{\mu}{\Gamma} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

where $\Gamma = \gamma + \lambda + \mu$.

With the help of the above "characteristic equations" we investigate the boundary spectrum of A in more detail. Since by Theorem 2.1 the semigroup is bounded, the spectral bound $\mathrm{s}(A)$ of A is not greater than 0. It indeed coincides with 0 as we can conclude from the following lemma:

Lemma 3.4. *0 is an eigenvalue of A, i.e.* $0 \in \sigma_p(A)$.

Proof. By Lemma 3.2 it suffices to prove that $1 \in \sigma_p(\Phi D_0)$. Since $\Phi D_0 : l^1 \longrightarrow l^1$, and

$$\Phi D_0 = \begin{pmatrix} \sum_{k=0}^{B} pq^k & \sum_{k=0}^{B-1} pq^k & \sum_{k=0}^{B-2} pq^k & \cdots & p+pq & p & 0 & 0 & \cdots \\ pq^{B+1} & pq^{B} & pq^{B-1} & \cdots & pq^2 & pq & p & 0 & \cdots \\ pq^{B+2} & pq^{B+1} & pq^{B} & \cdots & pq^3 & pq^2 & pq & p & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

where, $p = \frac{\mu}{\mu+\lambda}$, $q = \frac{\lambda}{\mu+\lambda}$.

The equation $\Phi D_0 c = c$ is equivalent to the following system of equations:

$$\begin{cases} \left(\sum_{k=0}^{B} pq^k\right) c_1 + \left(\sum_{k=0}^{B-1} pq^k\right) c_2 + \left(\sum_{k=0}^{B-2} pq^k\right) c_3 + \cdots + (p+pq)c_B + pc_{B+1} = c_1 \\ pq^{B+1}c_1 + pq^B c_2 + pq^{B-1}c_3 + \cdots + pq^2 c_B + pqc_{B+1} + pc_{B+2} = c_2 \\ pq^{B+2}c_1 + pq^{B+1}c_2 + pq^B c_3 + \cdots + pq^3 c_B + pq^2 c_{B+1} + pqc_{B+2} + pc_{B+3} = c_3 \\ pq^{B+3}c_1 + pq^{B+2}c_2 + pq^{B+1}c_3 + \cdots + pq^4 c_B + pq^3 c_{B+1} + pq^2 c_{B+2} + pqc_{B+3} + pc_{B+4} = c_4 \\ pq^{B+4}c_1 + pq^{B+3}c_2 + pq^{B+2}c_3 + \cdots + pq^5 c_B + pq^4 c_{B+1} + pq^3 c_{B+2} \\ \quad + pq^2 c_{B+3} + pqc_{B+4} + pc_{B+5} = c_5 \\ pq^{B+5}c_1 + pq^{B+4}c_2 + pq^{B+3}c_3 + \cdots + pq^6 c_B + pq^5 c_{B+1} + pq^4 c_{B+2} \\ \quad + pq^3 c_{B+3} + pq^2 c_{B+4} + pqc_{B+5} + pc_{B+6} = c_6 \\ \quad \cdots . \end{cases}$$

From this system of equations we obtain

$$c_{B+n+1} = \frac{c_{n+1} - qc_n}{1-q}, \quad n \geq 2. \tag{$*$}$$

We now define the function

$$f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto f(x) := q^{(B+1)x} - q^{(B+1)x+1} - q^x + q.$$

Clearly, f is continuously differentiable and

$$f'(x) = (B+1)(1-q)\ln q \mathrm{e}^{(B+1)x\ln q} - \ln q \mathrm{e}^{x\ln q}.$$

Since the traffic intensity $\rho = \frac{\lambda}{\mu} < 1$, it follows that $q = \frac{\lambda}{\mu+\lambda} < \frac{1}{2}$ and thus $(B+1)(1-q) > 1$. Hence we can estimate

$$f'(0) = (B+1)(1-q)\ln q - \ln q < 0$$

therefore there exists $x_0 > 0$ such that $f'(x) < 0$ for all $x \in (0, x_0)$, this means that $f(x)$ is decreasing on $(0, x_0)$, since $\lim_{x\to+\infty} f(x) = q$, $f(0) = 0$ and $f(x) < 0$ for all $x \in (0, x_0)$, hence there exists $a \in \mathbb{R}$ such that $f(a) = 0$. We take $c_n = q^{na}, n \geq 2$ and substitute $c_n = q^{na}, n \geq 2$ into the Eq. $(*)$, then

$$\begin{aligned}
&q^{(B+n+1)a} = \frac{q^{(n+1)a} - qq^{na}}{1-q} \\
&\Longrightarrow \\
&q^{(B+1)a} - q^{(B+1)a+1} - q^a + q = 0
\end{aligned}$$

i.e., $c_n = q^{na}, n \geq 2$ is a solution of the Eq. $(*)$. Substituting $c_n = q^{na}, n \geq 2$ into the first equation in the above system of equations we obtain that

$$\begin{aligned}
q^{B+1}c_1 &= \left(\sum_{k=0}^{B-1} pq^k\right) q^{2a} + \left(\sum_{k=0}^{B-2} pq^k\right) q^{3a} + \cdots + (p+pq)q^{Ba} + pq^{(B+1)a} \\
&= pq^{2a}\sum_{k=0}^{B-1} q^k + pq^{3a}\sum_{k=0}^{B-2} q^k + \cdots + (p+pq)q^{Ba} + pq^{(B+1)a} \\
&= pq^{2a}\frac{1-q^B}{1-q} + pq^{3a}\frac{1-q^{B-1}}{1-q} + \cdots + pq^{Ba}\frac{1-q^2}{1-q} + pq^{(B+1)a} \\
&= q^{2a}(1-q^B) + q^{3a}(1-q^{B-1}) + \cdots + q^{Ba}(1-q^2) + q^{(B+1)a}(1-q) \\
&= q^{2a}(1+q^a+q^{2a}+\cdots+q^{(B-1)a}) - q^{2a+B}(1+q^{a-1}+q^{2(a-1)}+\cdots+q^{(B-1)(a-1)}) \\
&= q^{2a}\left[\frac{1-q^{Ba}}{1-q^a} - q^B\frac{1-q^{B(a-1)}}{1-q^{(a-1)}}\right] \\
&= q^{2a}\frac{(1-q^{Ba})(1-q^{a-1}) - (q^B-q^{Ba})(1-q^a)}{(1-q^a)(1-q^{a-1})} \\
&\Longrightarrow
\end{aligned}$$

$$c_1 = q^{2a-B-1}\frac{(1-q^{Ba})(1-q^{a-1}) - (q^B-q^{Ba})(1-q^a)}{(1-q^a)(1-q^{a-1})}.$$

Since

$$\begin{aligned}
\sum_{n=1}^{\infty}|c_n| &= q^{2a-B-1}\frac{(1-q^{Ba})(1-q^{a-1}) - (q^B-q^{Ba})(1-q^a)}{(1-q^a)(1-q^{a-1})} + \sum_{n=2}^{\infty} q^{na} \\
&= q^{2a-B-1}\frac{(1-q^{Ba})(1-q^{a-1}) - (q^B-q^{Ba})(1-q^a)}{(1-q^a)(1-q^{a-1})} + q^{2a}(1+q^a+q^{2a}+\cdots) \\
&= q^{2a-B-1}\frac{(1-q^{Ba})(1-q^{a-1}) - (q^B-q^{Ba})(1-q^a)}{(1-q^a)(1-q^{a-1})} + \frac{q^{2a}}{1-q^a} < +\infty.
\end{aligned}$$

Obviously, $c = (c_1, c_2, c_3, \ldots) \in l^1$ and c is a fixed point of the operator ΦD_0, it follows from this that $1 \in \sigma_p(\Phi D_0)$. By Lemma 3.2 we conclude that $0 \in \sigma_p(A)$. $\square$

0 is the only spectral value of A on the imaginary axis as the following lemma shows.

Lemma 3.5.

$$\sigma(A) \cap i\mathbb{R} = \{0\}.$$

Proof. Since for any $\gamma = ai$, $a \in i\mathbb{R}$, $a \neq 0$, we have

$$\Phi D_\gamma = \begin{pmatrix} \frac{\mu\lambda}{(\lambda+\mu)\Gamma} + \sum_{k=1}^{B} \frac{\mu\lambda^k}{\Gamma^{k+1}} & \sum_{k=0}^{B-1} \frac{\mu\lambda^k}{\Gamma^{k+1}} & \sum_{k=0}^{B-2} \frac{\mu\lambda^k}{\Gamma^{k+1}} & \cdots & \frac{\mu}{\Gamma} + \frac{\mu\lambda}{\Gamma^2} & \frac{\mu}{\Gamma} & 0 & 0 & \cdots \\ \frac{\mu\lambda^{B+1}}{\Gamma^{B+2}} & \frac{\mu\lambda^{B}}{\Gamma^{B+1}} & \frac{\mu\lambda^{B-1}}{\Gamma^{B}} & \cdots & \frac{\mu\lambda^2}{\Gamma^3} & \frac{\mu\lambda}{\Gamma^2} & \frac{\mu}{\Gamma} & 0 & \cdots \\ \frac{\mu\lambda^{B+2}}{\Gamma^{B+3}} & \frac{\mu\lambda^{B+1}}{\Gamma^{B+2}} & \frac{\mu\lambda^{B}}{\Gamma^{B+1}} & \cdots & \frac{\mu\lambda^3}{\Gamma^4} & \frac{\mu\lambda^2}{\Gamma^3} & \frac{\mu\lambda}{\Gamma^2} & \frac{\mu}{\Gamma} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

where $\Gamma = \gamma + \lambda + \mu$ and for $j \geq B+1$ we have

$$\begin{aligned} \sum_{i=1}^{\infty} |a_{ij}| &= \left|\frac{\mu}{\Gamma}\right| + \left|\frac{\mu\lambda}{\Gamma^2}\right| + \left|\frac{\mu\lambda^2}{\Gamma^3}\right| + \left|\frac{\mu\lambda^3}{\Gamma^4}\right| + \cdots = \frac{\mu}{|\Gamma|} \sum_{k=0}^{\infty} \left(\frac{\lambda}{|\Gamma|}\right)^k \\ &= \frac{\mu}{|\Gamma|} \times \frac{1}{1 - \frac{\lambda}{|\Gamma|}}\Bigg| = \frac{\mu}{|\Gamma| - \lambda} < 1. \end{aligned}$$

For $1 \leq j < B+1$,

$$\begin{aligned} \sum_{i=1}^{\infty} |a_{i1}| &\leq \left|\frac{\mu\lambda}{(\lambda+\mu)\Gamma}\right| + \sum_{k=1}^{B} \left|\frac{\mu\lambda^k}{\Gamma^{k+1}}\right| + \left|\frac{\mu\lambda^{B+1}}{\Gamma^{B+2}}\right| + \left|\frac{\mu\lambda^{B+2}}{\Gamma^{B+3}}\right| + \cdots \\ &= \frac{\mu\lambda}{|\lambda+\mu||\Gamma|} + \frac{\mu}{|\Gamma|} \sum_{k=1}^{\infty} \left(\frac{\lambda}{|\Gamma|}\right)^k < \frac{\mu}{|\Gamma|} \sum_{k=0}^{\infty} \left(\frac{\lambda}{|\Gamma|}\right)^k < 1 \\ \sum_{i=1}^{\infty} |a_{ij}| &\leq \left|\frac{\mu}{\Gamma}\right| + \left|\frac{\mu\lambda}{\Gamma^2}\right| + \left|\frac{\mu\lambda^2}{\Gamma^3}\right| + \left|\frac{\mu\lambda^3}{\Gamma^4}\right| + \cdots = \frac{\mu}{|\Gamma|} \sum_{k=0}^{\infty} \left(\frac{\lambda}{|\Gamma|}\right)^k \\ &= \frac{\mu}{|\Gamma|} \times \frac{1}{1 - \frac{\lambda}{|\Gamma|}} = \frac{\mu}{|\Gamma| - \lambda} < 1. \end{aligned}$$

Therefore $\|\Phi D_\gamma\| < 1$ for all $\gamma = ai$, $a \in R$, $a \neq 0$, namely $\|r(\Phi D_\gamma)\| \leq \|\Phi D_\gamma\| < 1$, it follows from this that $1 \notin \sigma(\Phi D_\gamma)$. By Lemma 3.2 we obtain that $\gamma \notin \sigma(A)$, i.e. $\sigma(A) \cap iR = \{0\}$. □

In the next step we determine the resolvent of A in terms of the resolvent of A_0, the Dirichlet operator D_γ and the boundary operator Φ.

Lemma 3.6. *For any $\gamma \in \rho(A_0) \cap \rho(A)$ one has*

$$R(\gamma, A) = R(\gamma, A_0) + D_\gamma(\mathrm{Id} - \Phi D_\gamma)^{-1} \Phi R(\gamma, A_0).$$

Proof. If $\gamma \in \rho(A_0) \cap \rho(A)$ then by Lemma 3.2 we have $1 \in \rho(\Phi D_\gamma) \cap \rho(D_\gamma \Phi)$. Hence, we can compute

$$\begin{aligned} \mathrm{Id} &= \mathrm{Id} - D_\gamma \Phi + D_\gamma \Phi \\ &= \mathrm{Id} - D_\gamma \Phi + D_\gamma(\mathrm{Id} - \Phi D_\gamma)(\mathrm{Id} - \Phi D_\gamma)^{-1} \Phi \\ &= \mathrm{Id} - D_\gamma \Phi + D_\gamma(\mathrm{Id} - \Phi D_\gamma)^{-1} \Phi - D_\gamma \Phi D_\gamma(\mathrm{Id} - \Phi D_\gamma)^{-1} \Phi \\ &= \mathrm{Id} - D_\gamma \Phi + (\mathrm{Id} - D_\gamma \Phi) D_\gamma(\mathrm{Id} - \Phi D_\gamma)^{-1} \Phi. \end{aligned}$$

Multiplying both sides by $(\mathrm{Id} - D_\gamma \Phi)^{-1}$ yields

$$(\mathrm{Id} - D_\gamma \Phi)^{-1} = \mathrm{Id} + D_\gamma (\mathrm{Id} - \Phi D_\gamma)^{-1} \Phi. \tag{8}$$

By (7) the resolvent of A in γ is given by

$$R(\gamma, A) = (\mathrm{Id} - D_\gamma \Phi)^{-1} R(\gamma, A_0). \tag{9}$$

Putting (8) in the formula (9) for the resolvent we finally obtain

$$R(\gamma, A) = R(\gamma, A_0) + D_\gamma (\mathrm{Id} - \Phi D_\gamma)^{-1} \Phi R(\gamma, A_0). \quad \square$$

We can compute the resolvent of A_0 explicitly applying the formula for the inverse of operator matrices, see [6, Thm. 2.4].

For $\gamma \in \{\gamma \in \mathbb{C} | \gamma \neq -\lambda \text{ and } \mathrm{Re}\,\gamma > -(\lambda + \mu)\}$ the resolvent of A_0 is obtained as

$$R(\gamma, A_0) = \begin{pmatrix} \frac{1}{\gamma+\lambda} & \frac{\mu}{\gamma+\lambda} \int_0^\infty R(\gamma, D).\mathrm{d}x & 0 & 0 & \cdots \\ 0 & R(\gamma, D) & 0 & 0 & \cdots \\ 0 & \lambda R^2(\gamma, D) & R(\gamma, D) & 0 & \cdots \\ 0 & (\lambda)^2 R^3(\gamma, D) & \lambda R(\gamma, D) & R(\gamma, D) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

where

$$D := -\frac{\mathrm{d}}{\mathrm{d}x} - (\lambda + \mu), \quad (R(\gamma, D)p)(x) = \mathrm{e}^{-(\gamma+\lambda+\mu)x} \int_0^x \mathrm{e}^{(\gamma+\lambda+\mu)s} p(s)\mathrm{d}s$$

for $p \in L^1[0, \infty)$.

The above representation for the resolvent in particular shows that it is a positive operator for $\gamma > 0$. We need this property in the following lemma to prove the irreducibility of the semigroup generated by A:

Lemma 3.7. *The semigroup $(T(t))_{t\geq 0}$ generated by $(A, D(A))$ is irreducible.*

Proof. It suffices to show that there exists $\gamma > 0$ such that $0 \lneq p \in X$ implies $R(\gamma, A)p \gg 0$, see [7, Def. C-III 3.1]. By Lemma 3.6 we have to prove that there exists $\gamma > 0$ such that $0 \lneq p \in X$ implies

$$R(\gamma, A_0)p + D_\gamma (\mathrm{Id}_{\partial X} - \Phi D_\gamma)^{-1} \Phi R(\gamma, A_0)p \gg 0.$$

Suppose that $\gamma > 0$ and $0 \lneq p \in X$. Then also $R(\gamma, A_0)p \gneq 0$ and $\Phi R(\gamma, A_0)p \gneq 0$. Since $\|\Phi D_\gamma\| < 1$ for any $\gamma > 0$, the inverse of $\mathrm{Id}_{\partial X} - \Phi D_\gamma$ is given by the Neumann series

$$(\mathrm{Id}_{\partial X} - \Phi D_\gamma)^{-1} = \sum_{n=0}^{\infty} (\Phi D_\gamma)^n.$$

We know from the form of ΦD_γ that for every $i = 1, 2, \ldots$ there exists $k \in \mathbb{N}$ such that the real number $((\Phi D_\gamma)^k \Phi R(\gamma, A_0)p)_i > 0$, i.e. $(\mathrm{Id}_{\partial X} - \Phi D_\gamma)^{-1} \Phi R(\gamma, A_0)p \gg 0$, and by the form of D_γ we have

$$D_\gamma (\mathrm{Id}_{\partial X} - \Phi D_\gamma)^{-1} \Phi R(\gamma, A_0)p \gg 0.$$

This implies

$$R(\gamma, A)p \gg 0.$$

Therefore the semigroup $(T(t))_{t\geq 0}$ is irreducible. $\square$

Lemma 3.8. *The set $\{T(t) : t \geq 0\} \subseteq \mathcal{L}(X)$ is relatively compact for the weak operator topology.*

Proof. Since $0 \in \sigma_p(A)$, by [8, Cor. IV. 3.8] we know that there exists $0 \neq p \in \operatorname{fix}(T(t))_{t\geq 0}$. By the positivity of the semigroup we have

$$|p| = |T(t)p| \leq T(t)|p| \quad \text{for all } t \geq 0. \tag{10}$$

Suppose that $|p| < T(t)|p|$. By Theorem 2.1, $(T(t))_{t\geq 0}$ is a contraction semigroup and the norm on X is strictly monotone, we obtain

$$\|p\| < \|T(t)|p|\| \leq \|p\|$$

which is a contradiction, thus in (10),

$$|p| = T(t)|p|$$

holds and we can assume in the following without loss of generality that $p > 0$. Let $n \in \mathbb{N}$ and take $w \in [-nu, nu]$, i.e. $-nu \leq w \leq nu$, then

$$-np = -nT(t)p \leq T(t)w \leq nT(t)p = np \quad \text{for any } t \geq 0.$$

Since the order interval $[-np, np]$ is weakly compact in X, the orbit $\{T(t)w : t \geq 0\}$ is relatively weakly compact in X. Since $(T(t))_{t\geq 0}$ is irreducible, we obtain from [7, Prop. C-III 3.5 (a)] that p is a quasi-interior point of X which implies that

$$X_p := \bigcup_{n\geq 1} [-np, np]$$

is dense in X. We have shown that $\{T(t)w : t \geq 0\} \subset X_p$ and $\{T(t)w : t \geq 0\}$ is relatively weakly compact. Since the semigroup $(T(t))_{t\geq 0}$ is bounded, we know from [8, Lemma V. 2.7] that $\{T(t) : t \geq 0\} \subseteq \mathcal{L}(X)$ is relatively weakly compact. □

The mean ergodicity of the semigroup allows a decomposition of X into the direct sum of $ker A$ and $\overline{\operatorname{rg}(A)}$. Combining Lemmas 3.4, 3.5, 3.7 and 3.8 with Theorem 2.1 we obtain the following result:

Theorem 3.9. *X can be decomposed into the direct sum*

$$X = X_1 \oplus X_2$$

where $X_1 = \operatorname{fix}(T(t))_{t\geq 0} = \ker A$ is one-dimensional and spanned by a strictly positive eigenvector $\tilde{p} \in \ker A$ of A and $(T(t)|_{X_2})_{t\geq 0}$ is strongly stable.

Proof. By Lemma 3.7, $\{T(t) : t \geq 0\} \subseteq \mathcal{L}(X)$ is relatively weakly compact, hence we know from [8, Cor. V. 4.6] that $(T(t))_{t\geq 0}$ is mean ergodic, therefore it follows from [8, Lemma V. 4.6] that the space X can be decomposed into

$$X = \ker A \oplus \overline{\operatorname{rg}(A)} =: X_1 \oplus X_2$$

where $\ker A = \operatorname{fix}(T(t))_{t\geq 0}$. Since $0 \in \sigma_p(A)$, as in the proof of Lemma 3.8 we can show that there exists $\tilde{p} \in \ker A$ such that $\tilde{p} > 0$. Moreover, we find by the same construction as in the proof of [8, Lemma V. 2.20 (i)] $p' \in X'$ such that $p' > 0$ and $A'p' = 0$. Thus, by [7, Prop. C-III 3.5] we obtain that

$$\dim \ker A = 1$$

and that $\tilde{p}$ is strictly positive, i.e. $\tilde{p} \gg 0$.

Both spaces X_1 and X_2 are invariant under $(T(t))_{t\geq 0}$. Now we consider the restricted semigroup $(T_2(t))_{t\geq 0}$ where $T_2(t) := T(t)|_{X_2}$. Its generator $(A_2, D(A_2))$ is given by

$$A_2 v = Av, \quad D(A_2) = D(A) \cap X_2.$$

In the next step we show that $\sigma_p(A_2') \cap i\mathbb{R} = \emptyset$. From [8, Prop. IV. 1.12] we have

$$\sigma_p(A_2') = \sigma_r(A_2)$$

where $\sigma_r(A_2) = \{\gamma \in C : \operatorname{rg}(\gamma - A_2) \quad \text{is not dense in} \quad X_2\}$ denotes the residual spectrum. Since $\sigma(A_2) \subseteq \sigma(A)$ and $\sigma(A) \cap i\mathbb{R} = \{0\}$, we only have to prove that $0 \notin \sigma_p(A_2') = \sigma_r(A_2)$. Since $(T(t))_{t\geq 0}$ is a mean ergodic bounded semigroup on X, it follows from this that $(T_2(t))_{t\geq 0}$ is a mean ergodic bounded semigroup on X_2. By [8, Thm. V. 4.5], $\ker A_2$ separates $\ker A_2'$. But $\ker A_2 = \{0\}$, and thus $\ker A_2' = \{0\}$. It follows that $\sigma_p(A_2') = \emptyset$, i.e. $\sigma_p(A_2') \cap i\mathbb{R} = \emptyset$. By [8, Thm. 2.21], we obtain that $(T(t))_{t\geq 0}$ is strongly stable. $\square$

By Theorem 3.9 we obtain the main result in this paper.

Theorem 3.10. *For all $p \in X$ there exists $\alpha \in \mathbb{R}$ such that*

$$\lim_{t\to\infty} T(t)p = \alpha\tilde{p},$$

where $\ker A = \langle\tilde{p}\rangle$, $\tilde{p} \gg 0$.

Proof. By Theorem 3.9, we have

$$X = X_1 \oplus X_2$$

where $X_1 = \mathrm{fix}(T(t))_{t\geq 0} = \ker A$ is one-dimensional and spanned by a strictly positive eigenvector $\tilde{p} \in \ker A$, and $(T(t)|_{X_2})_{t\geq 0}$ is strongly stable, i. e. for any $p \in X$ there exists $\alpha \in \mathbb{R}$ and $p_2 \in X_2$ such that

$$p = \alpha\tilde{p} + p_2 \quad \text{and} \quad \lim_{t\to\infty} T(t)p_2 = 0.$$

Since $T(t)p = T(t)(\tilde{p} + p_2) = \alpha T(t)\tilde{p} + T(t)p_2 = \alpha\tilde{p} + T(t)p_2$, it follows that $\lim_{t\to\infty} T(t)p = \alpha\tilde{p}$. $\square$

By Theorem 3.10 we obtain asymptotic stability of the solution of $M/M^B/1$ model.

Theorem 3.11. *The time-dependent solution of the system (1)–(6) converges strongly to the steady-state solution as time tends to infinite, that is,* $\lim_{t\to\infty} p(\cdot, t) = \alpha\tilde{p}$, *where $\alpha > 0$ and $\tilde{p}$ as in Theorem 3.10.*

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